STATISTICAL THEORY OF COINCIDENCE EXPERIMENTS

TECHNICAL REPORT

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Electronics Research Project

THE GEORGE WASHINGTON UNIVERSITY

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Z. BAY, H. KANNER, V. P. HENRI

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ABSTRACT

The consistent use of a general statistical theory makes possible the elimination of ambiguities from the interpretation of coincidence experiments. It is shown that measurements of time delays and disintegration rates can be accomplished to any desired accuracy by use of experimental coincidence curves alone.

For the treatment of the time-resolving properties of coincidence equipment, two characteristic time magnitudes are needed, one of which is a straightforward generalization of the old definition of the resolving time. The two time magnitudes allow the approximate determination of random time lags. The general theory also provides a strict definition of the coincidence efficiency.

I. GENERAL CONSIDERATIONS

A consequence of recent improvements[1-9] in coincidence counting techniques is that a reformulation of some basic definitions has become necessary. For example, two hitherto equivalent definitions of the resolving time of a coincidence circuit are no longer equivalent or even, strictly speaking, meaningful. We shall demonstrate that a consistent and unambiguous set of definitions for all parameters needed in the interpretation of delayed coincidence measurements can be formulated in terms of experimentally observable quantities without any idealizing assumptions or approximations.

A usual method of obtaining coincidence curves is as follows: A source of pairs of particles is so placed that the members of each pair can enter two detectors. The output counting rate of the coincidence equipment including a discriminator, as a function of the delay time T inserted in one channel, is denoted by $N'(T)$. Then $N(T)$, the coincidence curve for the source in question is derived from the experimental curve by use of the relation

$$N(T) = N'(T) - N_c \qquad (1)$$

where N_c is the chance coincidence counting rate, given by $N_c = N'(\infty)$. If the two particles of each pair enter the detectors simultaneously, the resulting coincidence curve

is termed a "prompt" curve, and will be distinguished by the notation $F(T)$.

The height of the output pulse of the coincidence circuit for a given pair of input pulses will be a functional of the shapes of both input pulses as well as of their relative position in time. Since the detectors will in general emit pulses having a variety of shapes and will furthermore introduce randomly distributed time lags, it is apparent that the $F(T)$ curve will be descriptive of the entire system of detectors, coincidence circuit, and prompt source. The dependence on the source will hinge on the extent to which the type and energy of the particles will influence the pulse shape distributions.

Relating $N(T)$ and $F(T)$ to the same source strength, it is well known[10-14] that the connection between them is given by the convolution integral.

$$N(T) = \int_{-\infty}^{+\infty} F(T-t)\, p(t)\, dt \tag{2}$$

where $p(t)$ is the normalized probability density for the time interval t between entry of particles into the respective detectors. Eq. (2) is valid under the following conditions: a) the quantities t and T are interchangeable, i.e., the inserted delay mechanism does not materially affect the pulse shapes; b) the pulse shape

distributions in the respective channels are the same for both sources. Condition a) usually offers no experimental difficulty, as one generally uses short delay cables with negligible attenuation. Condition b) can be met either by assuring that the same type and energy of radiation enters the detectors from both sources,[9] or by eliminating the effect of any discrepancy between the radiations by proper pulse shaping techniques.[15]

As has been pointed out earlier,[13] a consequence of Eq. (2) is that the moments of N can be expressed in terms of those of F and p by the relations

$$M_n(N) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} M_{n-k}(F) M_k(p) \quad (3)$$

where $M_n(N) = \int_{-\infty}^{+\infty} T^n N(T) dT$, etc. This set of equations can be solved for the moments of p, thus determining the latter function. If p is known except for the values of a finite set of parameters, as many of equations (3) will be needed as there are parameters to be evaluated. It is convenient to rewrite Eq. (3) in terms of the normalized moments $\mu_n = M_n/M_o$ as

$$\mu_n(N) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \mu_{n-k}(F) \mu_k(P) \quad (3a)$$

since the normalized moments are independent of the source strengths.

Another parameter, the "total coincidence counting rate", to be denoted by N_o, will be needed for a clear understanding of the phenomena involved.[16] In particular, it is necessary for the definition and determination of resolving time and efficiency. We define a "coincidence-countable" pulse pair to be a pair of pulses, one from each detector, having the property that they will produce a coincidence count for some range or set of ranges of T of non-zero measure. The property that a pair of pulses be coincidence-countable is then dependent on the shapes of both pulses but independent of their relative time orientation. The quantity N_o is then defined to be the number of coincidence-countable pulse pairs which originate in unit time from related events at the source.

The description of the following thought experiment may serve to clarify the definition of N_o. Let a multi-channel delayed coincidence apparatus observe the pulses from two detectors, using a source of related events. By a multi-channel apparatus is meant a system wherein the detector outputs are branched and simultaneously observed by a number of delayed coincidence circuits, each having a different fixed value of T but being otherwise identical. Thus, a finite set of points spanning the delay curve is obtained in one measurement. Now, let the discriminator outputs of all the coincidence circuits go to one scaling circuit of sufficiently great

dead time that only one count will be registered for one source event, regardless of how many of the coincidence circuits respond to it. Then the counting rate registered by the scaler will asympototically approach N_o as the number of channels is increased indefinitely with a proportional decrease in ΔT, the interval between adjacent T values.

Practical methods for the direct experimental measurement of N_o have been devised;[17,18] the general principle will be described briefly in Section V and the methods presented in deta[illegible] in another paper.[15]

The theory can further be developed most simply in te[illegible]

$\phi(T) = F(T)/N_0$ and $\nu(T) = N(T)/N_o$ [illegible]

will be called "reduced coincidence curves." These [illegible] define the probability of a singl[illegible] coincidence-[illegible] producing a delayed coincide[illegible] cour[illegible] for a [illegible] terms of ϕ and ν, [illegible]q. (2) [illegible]

$$\nu(T) = \int_{-\infty}^{+\infty} \phi(T-t)\, p(t)\, dt$$

By [illegible]lying [illegible]

one [illegible]

$$\ldots(\nu) = M_0(q) \quad (4)$$

... coincidence curve is independent ... urthermore, since the coincidence ... time delays at the source from those ... ors or the coincidence device, the above ... can be ... nded to read: the area under a reduced coincidence curve is independent of all time delays, regardless of ... For ... iven coincidence circuit, this area depends only on ... shape distributions.

II. APPLICATION TO SIMPLE RADIOACTIVE DECAY

Application of Eq. (3a) to a simple parent-daughter decay is straightforward.[13] If one detector responds only to one of the particles, which case will be called "asymmetric" (as, for example, in a $\beta - \gamma$ experiment), the probability density for the elapsed time between the activation of the two detectors is:

$$p(t) = \frac{1}{\theta} e^{-t/\theta} \quad \text{for } t \geq 0$$
$$p(t) = 0 \quad \text{for } t < 0 \qquad (5)$$

Use of Eq. (3a) with n = 1 yields

$$\mu_1(N) = \theta + \mu_1(F) \qquad (6)$$

Thus, the mean life θ is given by the displacement of the centroids of the N and F curves.

In the event that both detectors have the same response to either particles, which case is termed "symmetric" (as is true in a γ-γ experiment with γ's of similar energy), the N(T) curve will have the same centroid as F(T). It is then necessary to use the second moments of the curves for the determination of θ. The above situation can be described by a symmetrized probability density:

$$p(t) = \frac{1}{2\theta}\, e^{-|t|/\theta} \tag{7}$$

In a coordinate system in which $\mu_1(F) = 0$, the application of Eq. (3a) for the second moment yields:

$$\mu_2(N) = \mu_2(F) + 2\,\theta^2 \tag{8}$$

The generalization of the theory to the analysis of the mean lives of a radioactive family will be given in Appendix B.

III. THE RESOLVING TIME

A careful analysis shows that two characteristic time magnitudes are needed to describe the "resolution" of a coincidence device,[16] and by their use the inconsistencies between several earlier definitions of the resolving time can be explained. The problems encountered in attempting to define the resolving time

of a coincidence circuit and the ultimate need for defining two distinct resolving times can best be appreciated by considering some simple cases.

First, let us assume that all pulses in a given channel are of the same shape and that no random time lags between events and pulses are introduced by the equipment. Then there will be a well-defined interval in the time coordinate describing the separation of the members of a pulse pair for which a coincidence will be recorded. The possibility that this interval consists of several separate subintervals will be disregarded here. (A rigorous treatment is given in Appendix A). The resolving time, τ , is customarily defined as half of the magnitude of this interval, and could be measured experimentally in two ways. One could either measure the coincidence curve for a prompt source, obtaining a rectangular curve of width 2τ , or one could observe the chance coincidence counting rate, N_c, for unrelated sequences of pulses in the two channels, obtaining

$$N_C = N_A N_B \cdot 2\tau \qquad (9)$$

where N_A and N_B are the respective "singles" conting rates.

Now, let us introduce one new feature: random time lags between the actual events and the pulses therefrom. The value of τ

will be unchanged, and can be determined as before from Eq. (9), since the chance coincidence counting rate will not be influenced by the random lags. However, the prompt coincidence curve will be broadened and will no longer be of rectangular shape. The maximum of the prompt curve will even be lowered in any of the random lags exceed τ. Despite the distortion of the coincidence curve, there is a convenient functional of it that will yield τ. Noting that in the absence of random time lags, the height of the rectangular coincidence curve will be N_0, the area of the curve divided by N_0 will be 2τ. Now, invoking the principle expressed by Eq. (4), that the area of a reduced coincidence curve is independent of time delays, we can write

$$\tau = \frac{1}{2N_0}\int_{-\infty}^{+\infty} F(T)\,dT = \frac{1}{2}\int_{-\infty}^{+\infty} \phi(T)\,dT \qquad (10)$$

also valid if $F(T)$ is replaced by $N(T)$, the coincidence curve for any source of related events, prompt or otherwise.

On the other hand, we see that the width of the coincidence curve is no longer given by the resolving time, τ. Nevertheless, the effective width of the prompt coincidence curve is indicative of the resolving power of the system for determination

of time delays. Therefore, it is desirable to define a second characteristic time, τ', that is a measure of this width. We choose the definition

$$\tau' = \frac{1}{2\phi_{max}} \int_{-\infty}^{+\infty} \phi(T)\, dT = \frac{1}{2F_{max}} \int_{-\infty}^{+\infty} F(T)\, dT \qquad (11)$$

Although this choice is somewhat arbitrary, it has the virtues that τ' reduces to τ in the absence of time lags, and that τ' is usually very close to the half-width at half maximum of the coincidence curve, being the half-width of the rectangle that has the same height and area as the curve.

The final step is to extend the definitions to the general case in which each channel receives a distribution of pulse shapes. Then a resolving time can be defined in the original manner for every possible kind of coincidence-countable pulse pair, and τ should be the average of these resolving times over the pulse shape distributions. We again consider first the case of no random time lags, and describe all possible coincidence-countable pulse pairs by a running index "i". Letting N_i be the number of "type i" pulse pairs appearing in unit time, the prompt coincidence curve will be given by

$$f(T) = \sum_i N_i f_i(T) \qquad (12)$$

where f_i is the coincidence curve for a single "type i" pulse pair, i.e., a rectangle of unit height and of the width $2\tau_i$. Without loss in generality, we can assume that all $f_i(T)$ have the same centroid, as failure to meet this condition can be compensated for by introduction of appropriate time lags. Then, since $\sum N_i = N_0$, the maximum of $f(T)$ will be N_0, and the average resolving time, τ, will be given by

$$\tau = \frac{\sum N_i \tau_i}{\sum N_i} = \frac{\frac{1}{2N_0} \sum N_i \int_{-\infty}^{+\infty} f_i(T)\,dT}{= \frac{1}{2N_0} \int_{-\infty}^{+\infty} f(T)\,dT} \tag{13}$$

as before, The argument already given for the case of uniform pulses again serves to extend Eq. (10) to apply to coincidence curves influenced by random time lags. Thus, 2τ is always given by the area of the reduced coincidence curve $\phi(T)$.

The chance coincidence counting rate will still be given by Eq. (9), provided the quantity $N_A N_B$ only includes coincidence-countable pulse pairs.

No further discussion of the definition of τ' is needed, since it is based on the shape alone of the observed prompt

coincidence curve and will be given in the general case also by Eq. (11). We choose to call τ the "true resolving time" and τ' the "practical resolving time". In Section VIII, we will show that the accuracy of mean life determinations is a function of the ratio of τ' to the mean life under measurement. Since $\phi_{max} \lesssim N_o$ we always have $\tau' \gtrsim \tau$.

IV. APPROXIMATE DETERMINATION OF THE AVERAGE RANDOM TIME LAG

The two experimentally measurable resolving times, τ' and τ, can be used to obtain the approximate magnitude of the random time lags occurring in the equipment.[16] We return to the formalism of Eq. (12), writing for a prompt coincidence curve with time lags

$$F(T) = \sum_i N_i \int_{-\infty}^{+\infty} f_i(T-t)\, \rho_i(t)\, dt \qquad (11.)$$

where $\rho_i(t)$ is a normalized probability density for the time lag t between the members of a "type i" pulse pair. Then the second moment of F(T) will be

$$M_2(F) = \sum_i N_i \left[M_2(\rho_i) M_0(f_i) + 2 M_1(\rho_i) M_1(f_i) + M_2(f_i) \right] \qquad (15)$$

The discussion will now be restricted to prompt curves obtained under symmetric conditions, i.e., identical detectors and similar radiation for the two channels. Experimental curves showing no detectable asymmetry have been obtained under such conditions. It is safe to assume then that all the ρ_i are symmetric functions of t, and that all f_i and F have a common centroid. Eq. (15) can therefore be rewritten:

$$M_2(F) = \sum_i N_i \left[M_2(\rho_i)\, M_0(f_i) + M_2(f_i) \right] \qquad (16)$$

Using the notation $f(T) = \sum N_i f_i(T)$ and $M_0(f_i) = 2\tau_i$ we have for the normalized moments

$$\mu_2(F) - \mu_2(f) = \frac{\sum N_i\, \mu_2(\rho_i) \cdot 2\tau_i}{\sum N_i \cdot 2\tau_i} \qquad (17)$$

Since F and f have a common centroid, the second moments on the left hand side of Eq. (17) can be replaced by second moments about the centroid. The expression on the right hand side is a weighted

average second moment of the random time lag with the weight factors τ_i .

We will now show that the left hand side of Eq. (17) can be replaced by $C\left[\tau'^2 - \tau^2\right]$, where C is a constant of the order of magnitude unity. To demonstrate this, we examine the ratio τ'/σ for a number of plausible functions, σ being the square root of the normalized second moment about the mean, and observe that this ratio is not sharply dependent on the shape of the function. For example, for a gaussian, $(\tau'/\sigma)^2 = \frac{\pi}{2}$; for an isoceles triangle, $(\tau'/\sigma)^2 = \frac{3}{2}$; and for the extreme case of a rectangle, $(\tau'/\sigma)^2 = 3$. Since $f_{max} = N_o$, there will be no distinction between τ' and τ for that function. Thus, with the assumption of a resonable similarity of shape between F and f, Eq. (17) can be written in the approximate form

$$2\bar{t}^2 = \tau'^2 - \tau^2 , \qquad (18)$$

$\bar{t}$ being the approximate rms time lag ascribed to one channel.

The validity of the approximations used in the derivation of Eq. (18) is demonstrated by the experimental data listed in

Table I. Using a fixed source of Co^{60} γ's and stilbene scintillators, τ alone was varied (by the differential coincidence method)[17] and the reduced coincidence curves taken. The values of τ' and τ were computed from the curves and $\bar{t}$ obtained using Eq. (18). It can be seen that $\bar{t}$ was essentially constant over the range of τ used.

Table I. Values of τ' and $\bar{t}$, for a number of values of τ, all other conditions remaining constant.

τ	τ'	$\bar{t}$
7.9×10^{-10} sec	14.5×10^{-10} sec	9.4×10^{-10} sec
5.6	14.7	9.6
3.1	13.5	9.3
1.65	13.5	9.2

We believe that the main contribution to $\bar{t}$ comes from the scintillation counters. Using Čerenkov counters,[19] values of $\bar{t}$ as low as 2.5×10^{-10} sec. have been observed, a value which can be attributed to the photomultipliers. For scintillators, theoretical limits have been calculated.[20, 21] A value of 3.1×10^{-10} sec. for $\bar{t}$ has been attained using diphenyl acetylene

crystals and a pulse shaping method.[22] This value lies within the theoretical estimates.

V. EXPERIMENTAL DETERMINATION OF N_o

Fig. 1 illustrates a simplified scheme for obtaining the coincidence curve $N(T)$ and the value of N_o appropriate to that curve. In the figure, C_1 represents the fast circuit that yields the coincidence curve; C_2 represents a similar circuit having a resolving time at least an order of magnitude greater than $\bar{\tau}$. The level of the discriminator following C_2 is set sufficiently high and that of C_1 sufficiently low (even tolerating noise pulses in C_1), that any pulse pair yielding an output from C_2 great enough to trigger C_2's discriminator is certain to produce an output from C_1's discriminator if the delay time, T, is correctly chosen. R represents a far slower coincidence circuit, the purpose of which is to determine whether a pair of outputs from C_1 and C_2 stem from the same source events. The output counting rate of R as a function of T will be the coincidence curve $N(T)$, and the counting rate of the discriminator following C_2 will be N_o.

Having determined N_o, one can use the reduced coincidence curve, which is normalized to one pair of events. Thus C_2 monitors the equipment and makes the delay curve independent of temporal

fluctuations. For example: 1) No correction is needed for the decrease in strength of a radioactive source while making a measurement; 2) the change of solid angle and thereby counting efficiency introduced by displacing the source, as is done in some time of flight measurements, does not affect the reduced coincidence curve.

VI. COINCIDENCE COUNTING EFFICIENCY

The chance coincidence counting rate was given in Eq. (9) as $N_A N_B \cdot 2\tau$, valid only if all pulse pairs are coincidence-countable. Eq. (9) can be written in terms of the disintegration rate, ν_0, of the source by introduction of ε_A and ε_B, the efficiencies for the counting of singles:

$$N_c = \nu_0^2 \varepsilon_A \varepsilon_B \cdot 2\tau \tag{19}$$

In practice, not all pulse pairs are coincidence-countable, and Eq. (19) must be modified to

$$N_C = \nu_0^2 \varepsilon_A \varepsilon_B \varepsilon_C \cdot 2\tau \tag{20}$$

where ε_c is a proportionality factor for ~~the fraction of~~ the pulse pairs ~~recorded as singles~~

that are coincidence-countable. Although it is common practice to regard a quantity such as $\mathcal{E}_C$ as the efficiency of the coincidence circuit proper, $\mathcal{E}_C$ has no independent physical meaning, since its value for a given circuit still depends on the values of $\mathcal{E}_A$ and $\mathcal{E}_B$, which in turn are determined by arbitrary discrimination levels. (In fact, it is possible to choose $\mathcal{E}_A$ and $\mathcal{E}_B$ in such a manner that $\mathcal{E}_C$ is greater than unity.) However, the product $\mathcal{E}_A\mathcal{E}_B\mathcal{E}_C$ is well defined, being the fraction of source events yielding coincidence-countable pulse pairs. This product, denoted by $\mathcal{E}$, will validly represent the overall efficiency of the coincidence equipment.

For the experimental determination of $\mathcal{E}$, we use the relation (See Eq. (A2) Appendix A)

$$N_0 = \nu_0 \mathcal{E} \tag{21}$$

By Eqs. (10), (20), and (21), $\mathcal{E}$ can be expressed in terms of experimentally observable quantities as

$$\mathcal{E} = \frac{N_0 \int_{-\infty}^{+\infty} N(T)\, dT}{N_c} \tag{22}$$

Eqs. (20 or (21) show that $\mathcal{E}$ is independent of time magnitudes such as $\bar{t}$ or τ'. While reduction of τ' alone lowers the counting rates along the coincidence curve, it does not reduce the coincidence counting efficiency. The decrease in counting rates is merely a consequence of an increased ability on the part of the coincidence circuit to discriminate against small time differences.

VII. MEASUREMENT OF DISINTEGRATION RATES

Coincidence methods have long been used for absolute measurement of source strengths,[23] their advantage being that they eliminate the need for knowledge of the detector efficiencies. With the use of coincidence devices of sufficiently great τ', the condition $N_{max} = N_o$ can be met, and by proper circuitry and choice of discrimination levels for the counting of singles, it is possible[2, 3] to have $\mathcal{E}_c = 1$. Under these conditions, the source strength will be given by

$$\nu_o = \frac{N_A N_B}{N_{max}} \tag{23}$$

This method is limited to relatively weak sources, for the chance coincidence counting rate must not be so high as to

prevent accurate measurement of N_{max}, and reduction of N_c by reducing τ cannot be extended indefinitely, since $N_{max} = N_o$ requires $\tau \gg \bar{t}$. Hence, the condition $\nu_o \ll \frac{1}{\bar{t}}$ sets an upper limit for applicability of this procedure.

Utilization of the observed chance coincidence counting rate provides another method for measurement of source strengths that is not subject to the above limitations and does not require measurement of the singles counting rates.[24] By combining Eqs. (10), (20), and (21), with the elimination of ε and τ, we obtain

$$\nu_o = \frac{N_c}{\int_{-\infty}^{+\infty} N(T)\,dT} \tag{24}$$

The observables on the right hand side of Eq. (24) are independent of random time lags, and the method can even be used for delayed sources having a mean life far greater than τ.

VIII. STATISTICAL ACCURACY OF MEAN LIFE DETERMINATIONS

In discussing the statistical accuracy of mean life measurements,[25] it is necessary to distinguish between the "symmetric" and "asymmetric" types of experiments. Equations (6) and (8)

respectively are used for the evaluation of θ in these two cases. The statistical accuracy of θ depends then on that of the normalized first or second moments of the coincidence curves used.

If ν (T) is regarded as the probability of obtaining a coincidence count per countable pulse pair, then N(T) may be regarded as the number of successes in N_o trials, with the a priori probability of success ν (T). The value of N(T) will then follow the binomial distribution

$$P(N) = C_N^{N_0} \nu^N (1-\nu)^{N_0 - N}, \tag{25}$$

the expected value of N(T) will be $N_o \nu$ (T), and the variance, σ^2, of N will be $N_o \nu (1- \nu)$. Using $N(T)/N_o$ as the experimental estimate of the probability ν (T), we have

$$\sigma^2(N) \simeq N(T)\left[1 - \frac{N(T)}{N_0}\right] \tag{26}$$

If the coincidence curve is obtained from a set of counting rates N_i taken at equidistant T values, μ can be approximated by $(1/\mathcal{N})\sum N_i T_i^n$, where $\mathcal{N} = \sum N_i$, the total number of counts observed along the entire coincidence curve. Employing the usual method of treatment of the propagation of errors,

$$\sigma^2(\mu_n) = \frac{1}{\mathcal{N}^2}\sum_i N_i\left(1-\frac{N_i}{N_0}\right)\left\{T_i^{2n} - 2\mu_n T_i^n + \mu_n^2\right\} \tag{27}$$

An upper limit for $\sigma^2(\mu_n)$ can then be obtained by omission of the factor $1 - N_i/N_o$:

$$\sigma^2(\mu_n) < \frac{1}{\mathcal{N}}\left(\mu_{2n} - \mu_n^2\right) . \tag{28}$$

In actual practice, the right hand side of (28) has been found to overestimate $\sigma^2(\mu_n)$ by only 10% for a typical coincidence curve.

The standard deviations of θ in terms of those of the moments used are

$$\sigma(\theta) = \left\{\sigma^2\left[\mu_1(N)\right] + \sigma^2\left[\mu_1(f)\right]\right\}^{1/2} \tag{29a}$$

for the asymmetric case, and

$$\sigma(\theta) = \frac{1}{4\theta}\left\{\sigma^2\left[\mu_2(N)\right] + \sigma^2\left[\mu_2(f)\right]\right\}^{1/2} \qquad (29b)$$

for the symmetric case.

One further approximation yields even simpler formulae, particularly useful for order of magnitude estimates when planning experiments. This consists of fitting the gaussian $C\exp\left(-\pi t^2/4\tau'^2\right)$ to the prompt coincidence curve F(T) and expressing all pertinent moments of F(T) and N(T) in terms of τ'/θ. The results of this procedure are:

$$\frac{\sigma(\theta)}{\theta} = \frac{1}{\sqrt{\pi}}\left\{1 + \frac{4}{\pi}\left(\frac{\tau'}{\theta}\right)^2\right\}^{1/2} \qquad (30a)$$

for the asymmetric case, and

$$\frac{\sigma(\theta)}{\theta} = \frac{1}{\sqrt{\pi}}\left\{\frac{5}{4} + \frac{1}{\pi}\left(\frac{\tau'}{\theta}\right)^2 + \frac{1}{\pi^2}\left(\frac{\tau'}{\theta}\right)^4\right\}^{1/2} \qquad (30b)$$

for the symmetric case.

IX. APPENDIX A

A formulation of the theory wherein the pulse shape distributions are explicitly introduced has several virtues. In particular, it yields a straightforward mathematical interpretation

of N_0, τ, $\mathcal{E}$, and of the counting rate for chance coincidences.

The pulse shapes will be described by a set of parameters α_K and β_K for those in channels A and B respectively. Using $V_A(t)$ for the voltage in channel A as a function of time, a possible (but obviously not unique) set of suitable parameters is:

$$\alpha_0 = \int V_A(t)\,dt$$

$$\alpha_K = \int \left(t - \langle t\rangle_{AV}\right)^k V_A(t)\,dt$$

where

$$\langle t\rangle_{AV} = \frac{1}{\alpha_0}\int t\, V_A(t)\,dt\,,$$

the centroid of the pulse. The relative orientation in time of a pulse pair can be described by

$$t_{AB} = \langle t_A\rangle_{AV} - \langle t_B\rangle_{AV}\,.$$

The pulse shape distributions in channels A and B respectively will be denoted by $P(\alpha_1, \alpha_2, \ldots)\,d\alpha_1\,d\alpha_2\ldots$ and $Q(\beta_1, \beta_2, \ldots)\,d\beta_1\,d\beta_2\ldots$ henceforth to be abbreviated as $P(\alpha)\,d\alpha$ and $Q(\beta)\,d\beta$. For related events, there will also be a distribution in t_{AB}, denoted by $R(t_{AB})\,dt_{AB}$, R being in general dependent also on the α's and β's.

Now the property that a given pulse pair produces or fails to produce a coincidence count is a function of the α's, β's, and t_{AB}. The α, β, t_{AB} space can thus be divided into two parts: that corresponding to pulse pairs which will produce a

count, and corresponding to those which will not. The former region is designated in Fig. 2 by $\mathcal{V}$. If a source produces ν_0 simultaneous event pairs in unit time, the coincidence counting rate will be

$$F(T) = \nu_0 \iiint\limits_{\mathcal{V}} P(\alpha)\, Q(\beta)\, R(t_{AB} - T)\, dt_{AB}\, d\alpha\, d\beta \qquad \text{(A1)}$$

where T, as before, is the inserted delay time, the effect of which is a pure translation of the volume $\mathcal{V}$ along the t_{AB} axis. Similarly, N_0, the total coincidence counting rate and $\mathcal{E}$, the overall efficiency desribed earlier, will be readily defined by

$$N_0 = \nu_0 \iint\limits_{\mathcal{A}} P(\alpha)\, Q(\beta)\, d\alpha\, d\beta = \nu_0 \mathcal{E} \qquad \text{(A2)}$$

where $\mathcal{A}$ is the projection of $\mathcal{V}$ on the α, β subspace.

For a given set of values of the α's and β's , denoted in the figure by α^0, β^0 , there will be a section of the line $\alpha = \alpha^0$, $\beta = \beta^0$, that lies within $\mathcal{V}$. This section

may have more than one part if the line intersects a cavity in $\mathcal{V}$. For example in Fig. 2, the portion of $\alpha = \alpha^0$, $\beta = \beta^0$, that lies within $\mathcal{V}$ runs from t_1 to t_2 and from t_3 to t_4. Equation (A1) can then be written:

$$F(T) = \nu_0 \iint_{\mathcal{Q}} P(\alpha) Q(\beta) \left[\sum_i \int_{\Delta_i t_{AB}} R(t_{AB} - T) \, dt_{AB} \right] d\alpha \, d\beta \qquad (A3)$$

where $\Delta_i t_{AB}$ denotes the i^{th} distinct portion of the above mentioned line falling within $\mathcal{V}$. The end points of each $\Delta_i t_{AB}$ interval will, of course, be functions of α and β.

If we now compute the area of the coincidence curve, we obtain

$$\int_{-\infty}^{+\infty} F(T) \, dT = \nu_0 \iint_{\mathcal{Q}} P(\alpha) Q(\beta) \cdot 2\tau(\alpha, \beta) \, d\alpha \, d\beta \qquad (A4)$$

where $2\tau(\alpha,\beta)$ is the measure of the section of the line corresponding to an α and β falling within $\mathcal{V}$. For the line shown in Fig. 2, $2\tau = t_4 - t_3 + t_2 - t_1$. τ as defined by (10), can now be written:

$$\tau = \frac{\int_{-\infty}^{+\infty} F(T) \, dT}{2 N_0} = \frac{\iint_{\mathcal{Q}} P(\alpha) Q(\beta) \tau(\alpha, \beta) \, d\alpha \, d\beta}{\iint_{\mathcal{Q}} P(\alpha) Q(\beta) \, d\alpha \, d\beta} \qquad (A5)$$

Thus, $\overline{T}$ can be interpreted as an average over the pulse shape distributions of $T(\alpha, \beta)$, the resolving time for pulses of shapes specified by α and β.

The chance coincidence counting rate can now be computed readily. Consider two independent sources of strengths ν_A and ν_B producing pulses in the respective channels. By a "type $-\alpha$" pulse is meant a pulse whose parameters are in a specified range $\alpha_1, \alpha_2, \ldots$ to $\alpha_1 + d\alpha_1, \alpha_2 + d\alpha_2$. For a single type $-\alpha$ pulse in A and a single type $-\beta$ pulse in B, the available interval on the t_{AB} axis for coincidence is $2T(\alpha, \beta)$. The number of type $-\beta$ pulses per unit time will be $\nu_B Q(\beta)\, d\beta$. Hence the total accumulated interval over unit time for coincidence between one type$-\alpha$ pulse and any pulse in B will be $\nu_B \int Q(\beta)\, 2T(\alpha, \beta)\, d\beta$. This interval divided by the unit of time will then be the probability of a coincidence between the type $-\alpha$ pulse and any type $-\beta$ pulse. The chance coincidence counting rate can therefore be obtained by multiplication by the number of type $-\alpha$ pulse and integration over α, i.e.,

$$N_C = \nu_A \nu_B \iint_{\mathcal{R}} P(\alpha)\, Q(\beta)\, 2T(\alpha, \beta)\, d\alpha\, d\beta\,. \qquad (A6)$$

The chance coincidence counting rate from one source of related events can now be determined by setting $\nu_A = \nu_B = \nu_0$:

$$N_c = \nu_0^2 \iint_{\mathcal{R}} P(\alpha)\, Q(\beta)\, 2\tau(\alpha,\beta)\, d\alpha\, d\beta\,, \qquad \text{(A7)}$$

or by the use of Eqs. (A2) and (A5):

$$N_c = \nu_0^2 \varepsilon \cdot 2\tau = \nu_0 N_0 \cdot 2\tau = \nu_0 \int_{-\infty}^{+\infty} N(T)\, dT$$

which is equivalent to Eq. (24).

X. APPENDIX B

To illustrate the application of Eqs. (2) and (2a) to more complicated cases, we will now treat the case of parent-daughter-granddaughter decay and will then make the obvious generalization to larger radioactive families.

Let us denote the mean life of the daughter by θ_1 and that of the granddaughter by θ_2. Without loss in generality, we can restrict the problem to the case in which only the radiation of the parent and granddaughter decays can excite the detectors.

The p(t) function for the time interval between the parent and granddaughter decays will be the superposition of the two exponential decay functions with the respective mean lives θ_1, and θ_2. The p(t) function and hence its moments will be symmetric in θ_1, and θ_2. The moments, μ_n, will be given by

$$\begin{aligned}
\mu_1 &= \theta_1 + \theta_2 = S_1 \\
\mu_2 &= 2\theta_1^2 + 2\theta_1\theta_2 + 2\theta_2^2 = 2S_1^2 - 2S_2 \\
\mu_3 &= 6\theta_1^3 + 6\theta_1^2\theta_2 + 6\theta_1\theta_2^2 + 6\theta_2^3 \\
&= 6S_1^3 - 12S_1S_2 \\
\mu_4 &= 24\theta_1^4 + 24\theta_1^3\theta_2 + 24\theta_1^2\theta_2^2 + 24\theta_1\theta_2^3 + 24\theta_2^4 \\
&= 24S_1^4 - 72S_1^2S_2 + 24S_2^2
\end{aligned} \tag{B1}$$

etc.

where $S_1 = \theta_1 + \theta_2$, $S_2 = \theta_1 \theta_2$ are the elementary symmetric functions of the θ's. Now, for an asymmetric experiment, the first moment of p(t) will yield S_1, the second moment and the known value of S_1 will yield S_2, and θ_1, and θ_2 will be the roots of the second degree equation

$$\theta^2 - S_1 \theta + S_2 = 0 \tag{B2}$$

The generalization to the analysis of an asymmetric experiment performed on a family of N members is straightforward. We denote the N elementary symmetric functions of the θ's by S_{kN}, $(K = 1 --- N)$. Since $\mu_n(P)$ is a symmetric function of the n-th degree in the θ's, μ_n can always be expressed as a function of the S_{kN} such that

$$\mu_n(P) = F_n(S_{1N}, ---, S_{nN}), \tag{B3}$$

where F_n does not contain any S_{kN} for $k > n$. Thus the first N moments of p utilized in ascending order will yield the S_{kN}'s by simple substitution, since the right hand side of Eqs. (B3) will be a triangular array in the S_{kN}'s. By a well-known theorem of algebra, $\theta_1, \theta_2, ---, \theta_N$ will be the roots of the N-th degree equation:

$$\sum_{K=0}^{n} (-1)^K S_{KN} \theta^{N-K} = 0 \tag{B4}$$

The p(t) function obtained in a symmetric experiment differs in that all the odd moments are zero. The even moments, however, will be the same as those obtained in an asymmetric experiment. Thus, for the parent-daughter-granddaughter case, S_1 and S_2 can be determined by inserting μ_2 and μ_4 in Eqs. (B1). While the previous triangular array is no longer available, Eq. (B2) is still valid and yields θ_1, and θ_2.

For the case of a symmetric experiment performed on a family of N members, we use the first N even moments $\mu_2, \mu_4, ---, \mu_{2N}$ from which we determine S_{kN}, $k = 1, --- N$. Then Eq. (B4) can be solved for $\theta_1, \theta_2, ---, \theta_N$ as before.

List of references:

[1] Z. Bay and G. Papp, Rev. Sci. Instr., 19, 565 (1948)

[2] R. E. Bell and H. E. Petch, Phys. Rev. 76, 1409 (1949)

[3] K. P. Meyer, E. Baldinger, and P. Huber, Helv. Phys. Acta 23, 121 (1950)

[4] R. Hofstadter and J. A. McIntyre, Phys. Rev. 78, 24 (1950)

[5] M. Deutsch and W. E. Wright, Phys. Rev. 77, 139 (1950); Phys. Rev. 82, 277 (1951)

[6] F. K. McGowen, Doctoral Thesis, Oak Ridge Report, ORNL 952 (1951).

[7] Z. Bay, Rev. Sci. Instr. 23, 397 (1951)

[8] S. De Benedetti, Rev. Sci. Instr. 23, 37 (1952).

[9] R. E. Bell, R. L. Graham, and H. E. Petch, Can J. Phys. 30, 35 (1952)

[10] H. Bradt and P. Scherrer, Helv. Phys. Acta 16, 251 (1943); 16, 259 (1943)

[11] F. W. Van Name, Phys. Rev. 75, 100 (1949)

[12] D. Binder, Phys. Rev. 76, 856 (1949)

[13] Z. Bay, Phys. Rev. 77, 419 (1950)

[14] T. D. Newton, Phys. Rev. 78, 490 (1950).

[15] Z. Bay, V. P. Henri, H. Kanner, and F. McLernon, to be published.

[16] Z. Bay, Phys. Rev. 87, 194 (1952).

[17] Z. Bay. Phys. Rev. 83, 242 (1951)

[18] Z. Bay, R. R. Meijer, and G. Papp, Nucleonics, 10, 38 (1952)

[19] Z. Bay, M. R. Cleland, and F. McLernon, Phys. Rev. 87, 901 (1952).

[20] R. F. Post, Nucleonics, 10, 46 (1952)

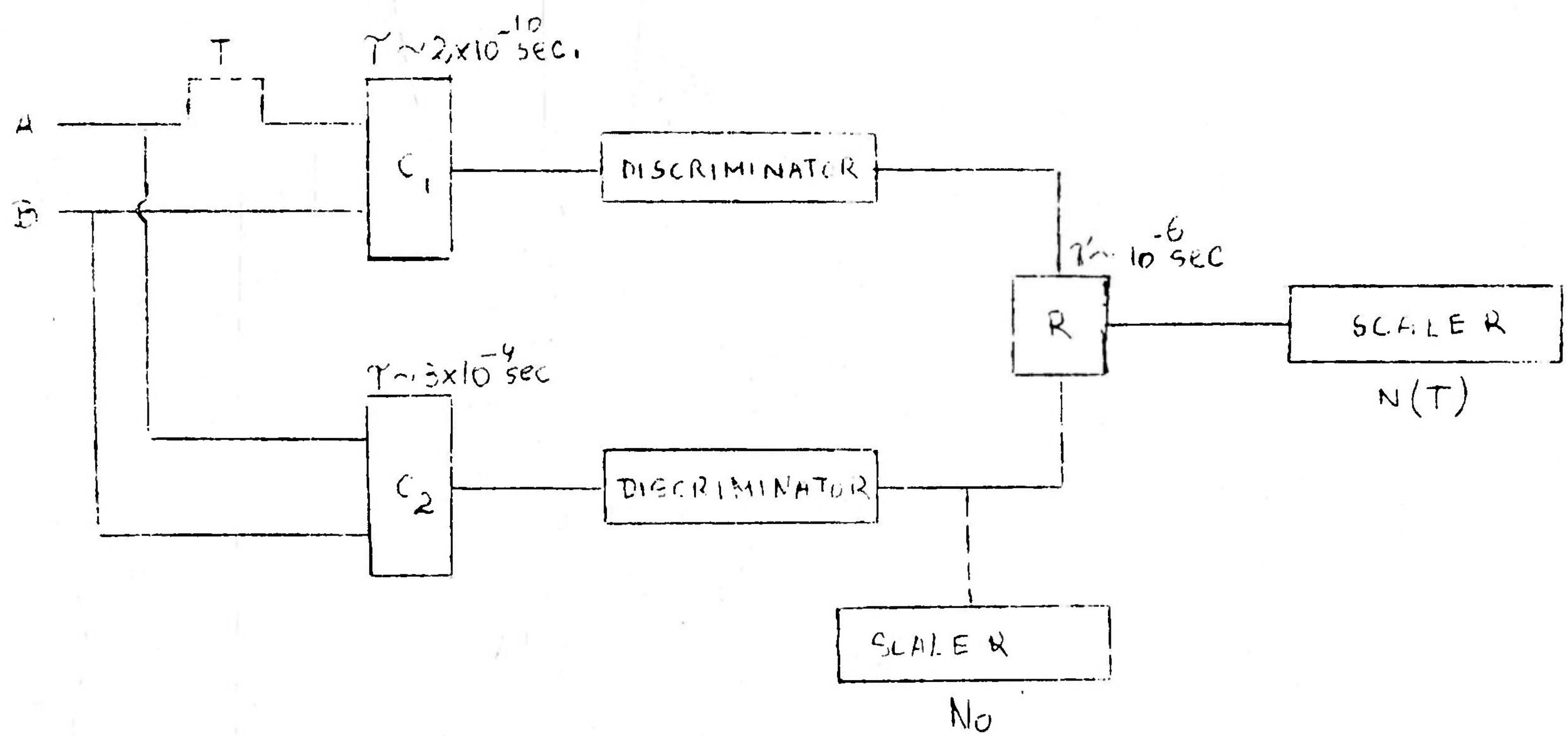
[21] R. F. Post and L. I. Schiff, Phys. Rev. 80, 1113 (1950)

[22] Z. Bay, V. P. Henri, and F. McLernon, Phys. Rev. 90, 371 (1953)

[23] Allan C. G. Mitchell, Rev. Mod. Phys. 20, 296 (1948)

[24] Z. Bay and M. M. Slawsky, Phys. Rev. 77, 414 (1950).

[25] H. Kanner, Z. Bay, and V. P. Henri, Phys. Rev. 90, 371 (1953).



A
B
T
$\tau \sim 2 \times 10^{-10}$ sec.
C_1
DISCRIMINATOR
$\tau \sim 3 \times 10^{-4}$ sec
C_2
DISCRIMINATOR
$\tau \sim 10^{-6}$ sec
R
SCALER
N(T)
SCALER
N_0

Fig. 1

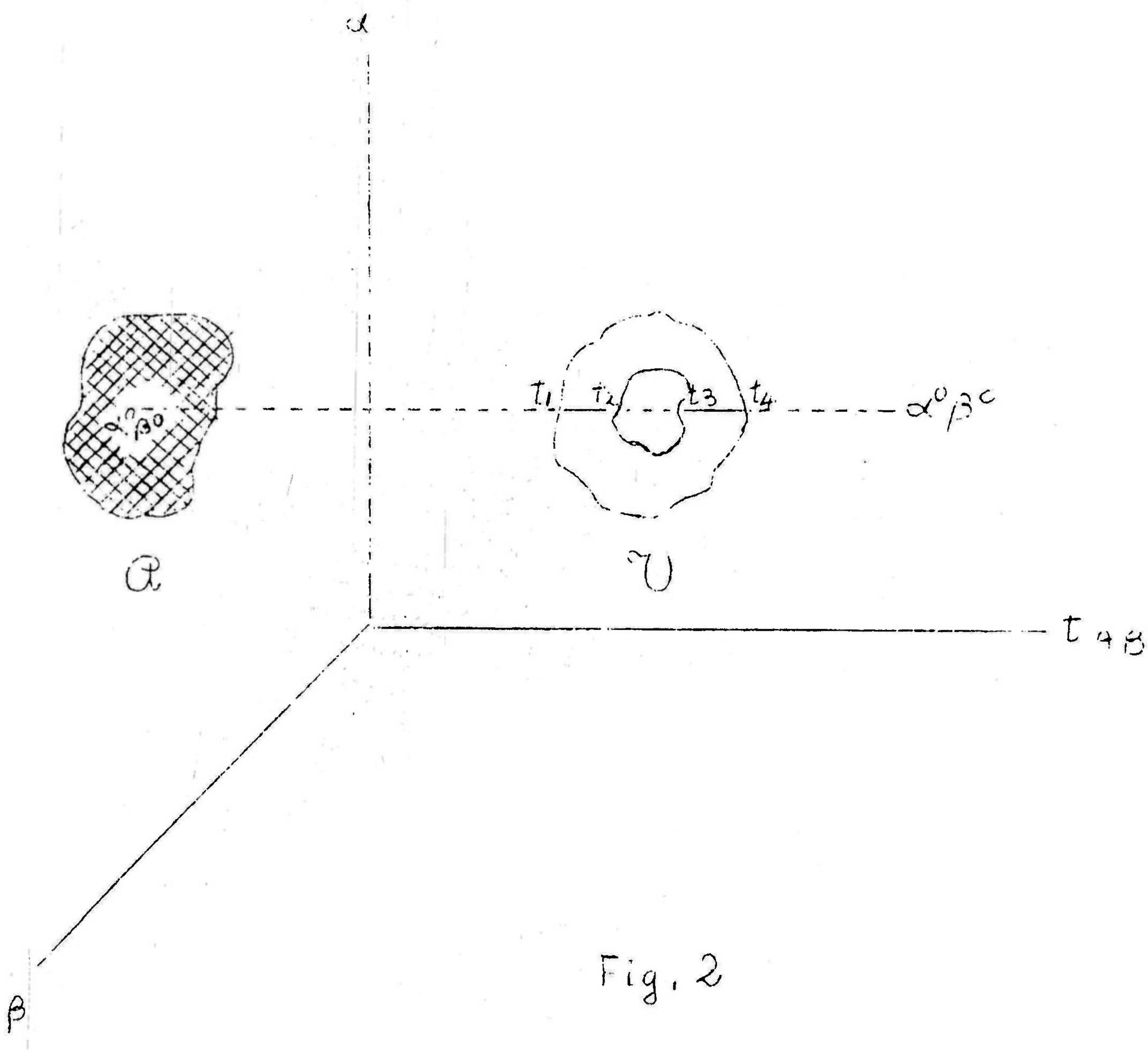

α
$\alpha^0 \beta^0$
$\mathcal{A}$
t_1 t_2 t_3 t_4
$\alpha^0 \beta^c$
$\mathcal{V}$
t_{4B}
β

Fig. 2